

# Higher order transverse bundles and riemannian foliations

Paul POPESCU

## Abstract

The purpose of this paper is to prove that each of the following conditions is equivalent to that the foliation  $\mathcal{F}$  is riemannian: 1) the lifted foliation  $\mathcal{F}^r$  on the  $r$ -transverse bundle  $\nu^r\mathcal{F}$  is riemannian for an  $r \geq 1$ ; 2) the foliation  $\mathcal{F}_0^r$  on a slashed  $\nu_*^r\mathcal{F}$  is riemannian and vertically exact for an  $r \geq 1$ ; 3) there is a positively admissible transverse lagrangian on a  $\nu_*^r\mathcal{F}$ , for an  $r \geq 1$ . Analogous results have been proved previously for normal jet vector bundles.

## 1 Introduction

Most of geometrical objects considered on a differentiable manifold using the tangent bundle can be constructed on a foliated manifold using the normal bundle. This is described in an algebraic manner, using natural functors, as in [17]. We use in this paper the normal bundle of order  $r$ , a foliated bundle that is a counterpart of the holonomy invariant of the tangent bundle of order  $r$ .

Various conditions that a foliation be riemannian are studied in many papers, for example [4, 5, 10, 12, 13, 16].

The conditions studied in this paper are closely related to [4, 5, 10, 12] and they have initially the origin in a special case of a problem presented by E. Ghys in Appendix E of P. Molino's book [6], i.e. asking if the existence of a foliated Finsler metric assure that a foliation is riemannian (see [4, 5, 12, 10] for more details). According to [10], the answer is affirmative in a more general case of a transverse lagrangian fulfilling a natural regularity condition, automatically fulfilled by a transverse Finslerian. The case of a transverse Finslerian on a compact manifold is studied in [4], using a different method.

The conditions in [10, 12] involve the existence of suitable admissible lagrangians or foliated metrics on the normal jet vector bundles. The aim of this paper is to study similar problems as in [12], but on higher order normal bundles. We use basic notions about foliations from [6, 17] and some notions relating foliated bundles and a basic result stated in Proposition 3.1 from [12]. We use some basic notions on higher order tangent spaces from [3, 7, 8]; we point some differences used here (for example, the lift of sections used here is

different from that used in [8] and the canonical inclusions in Proposition 2.1). Deeper properties of foliated hamiltonians are studied in [14]; it can be a model in a further study of some geometric objects considered in our paper.

The first goal of this paper is to find conditions that a foliation be riemannian, involving general conditions on higher order normal bundles. But some other aspects of the problem can be stressed. For example, the leaves of a riemannian foliation  $\mathcal{F}$  are compact, then the leaf spaces  $M/\mathcal{F}$  is a Satake manifold (or a V-manifold, in the original terminology of Satake), one of the first known non-trivial orbifold. The existence of a transverse lagrangian or hamiltonian is worth to be studied on such generalized manifolds, together with their physical properties; it is also the case of the normal bundle of a foliation studied in [10]. By Proposition 2.4 below, the existence of a foliated regular lagrangian of order  $r$ , that has a positively defined hessian, gives rise to a transverse riemannian metric on the normal bundle of order  $r + 1$  (see [3] for the non-foliated case). Also, a transverse metric of a riemannian foliation lifts to a transverse metric to the lifted foliation on the normal bundle of order  $r$ , that becomes a riemannian foliation (see Proposition 2.5 for a simple construction). Thus it is natural to consider the following problem: *under which conditions the existence of a regular and positive lagrangian or of a transverse metric on the normal bundle of a higher order transverse foliation assures that given foliation is riemannian?* A similar problem is studied for normal jet vector bundles (or  $(p, r)$ -velocities according to [16, 17]) in [12].

The first main result proved in this paper asserts that *the lifted foliation on a normal bundle of some order  $r \geq 1$  is riemannian iff the given foliation is riemannian* (Theorem 3.1). A simple consequence is that a foliation  $\mathcal{F}$  is a riemannian one, provided that the lifted foliation  $\mathcal{F}^r$  on  $\nu^r \mathcal{F}$  is transversally parallelizable or almost parallelizable. The proof of Theorem 3.1 can not give any answer to the following question: *when is  $\mathcal{F}$  riemannian if the foliation induced on  $\nu^r \mathcal{F} \setminus I_{r-1}^r(\nu^{r-1} \mathcal{F})$  is riemannian for some  $r \geq 1$ ?* We give some answers to this question, as follows. *The lifted foliation  $F_0^r$  on a suitable slashed bundle  $\nu_*^r F$  of the  $r$ -normal bundle  $\nu^r F$  is riemannian and vertically exact for some  $r \geq 1$  iff  $F$  is riemannian* (Theorem 3.2). In Theorem 3.3 we prove that *there is a positively admissible lagrangian on  $\nu^r F$  for some  $r \geq 1$  iff the foliation  $F$  is riemannian*. These three Theorems are analogous to [12, Theorems 1.1, 1.2 and 1.3], proved in the case of normal jet vector bundles.

As a conclusion, the results in this paper, together with that proved in [12] for normal jet vector bundles, confirm that asking some suitable natural conditions on a higher order lagrangian, the given foliation is necessarily riemannian; thus riemannian foliations are necessary setting to study certain transverse lagrangians, subject to natural conditions, considered on normal jet vector bundles or on higher order normal bundles of a foliation.

## 2 Basic notions and constructions related to the higher order transverse bundles of a foliation

Let  $M$  be an  $n$ -dimensional manifold and  $\mathcal{F}$  be a  $k$ -dimensional foliation on  $M$ . We denote by  $\tau\mathcal{F}$  and  $\nu\mathcal{F}$  the tangent plane field and the normal bundle respectively. A bundle  $E$  over  $M$  is called *foliated* if there is an atlas of local trivialisations on  $E$  such that all the components of the structural functions are basic ones (see, for example, [17]). In this case a canonical foliation  $\mathcal{F}_E$  on  $E$  is induced, having the same dimension  $k$ , such that  $p$  restricted to leaves is a local diffeomorphism. In particular, we consider affine and vector bundles that are foliated. For example,  $\nu\mathcal{F}$  is a foliated bundle and a natural foliation on  $\nu\mathcal{F}$  can be considered. According to [12], a *positively admissible lagrangian* on a foliated vector bundle  $p : E \rightarrow M$  is a continuous map  $L : E \rightarrow \mathbb{R}$  that is asked to be differentiable at least when it is restricted to the total space of the slashed bundle  $E_* = E \setminus \{\bar{0}\} \rightarrow M$ , where  $\{\bar{0}\}$  is the image of the null section, such that the following conditions hold: 1)  $L$  is positively defined (i.e. its vertical hessian is positively defined) and  $L(x, y) \geq 0 = L(x, 0)$ ,  $(\forall)x \in M$  and  $y \in E_x = p^{-1}(x)$ ; 2)  $L$  is locally projectable on a transverse lagrangian; 3) there is a basic function  $\varphi : M \rightarrow (0, \infty)$ , such that for every  $x \in M$  there is  $y \in E_x$  such that  $L(x, y) = \varphi(x)$ . If a positively transverse lagrangian  $F$  is 2-homogeneous (i.e.  $F(x, \lambda y) = \lambda^2 F(x, y)$ ,  $(\forall)\lambda > 0$ ), then  $F$  is called a *finslerian*; it is also a positively admissible lagrangian, taking  $\varphi \equiv 1$ , or any positive constant. For a foliated bundle, we can regard the vertical bundle  $VTE = \ker p_* \rightarrow E$  as a vector subbundle of  $\nu\mathcal{F}_E \rightarrow E$  by mean of the canonical projection  $TE \rightarrow \nu\mathcal{F}_E$ , since  $VTE$  is transverse to  $\tau\mathcal{F}_E$ . Notice that if  $p : E \rightarrow M$  is an affine bundle, then the vertical hessian  $\text{Hess } L$  of a lagrangian  $L : E \rightarrow \mathbb{R}$  is a symmetric bilinear form on the fibers of the vertical bundle  $VTE$ , given by the second order derivatives of  $L$ , using the fiber coordinates (see [10] for more details using coordinates).

In order to have a unitary form of the notions we use, we give now the constructions of the higher order normal bundle of a foliation and the related geometric objects used in the paper. They are similar to the non-foliate (or foliated by points) case as in [3, 7, 8], some are used for example in [15] in the foliate case.

Let us suppose that a foliation  $\mathcal{F}$  is defined by a foliated atlas having the generic coordinates  $(x^u, x^{\bar{u}})_{u=\overline{1,p}, \bar{u}=\overline{1,q}}$  on a chart  $(U, \varphi)$ , where  $p$  is the dimension of the leaves and  $q$  is the transverse dimension. The local form of the pseudogroup according to the coordinates change, is  $x^{u'} = x^{u'}(x^u, x^{\bar{u}})$ ,  $x^{\bar{u}'} = x^{\bar{u}'}(x^u, x^{\bar{u}})$ ; here  $(x^{u'}, x^{\bar{u}'})$  are some coordinates in a chart  $(U', \varphi')$ ,  $U \cap U' \neq \emptyset$ , and the pseudogroup member is  $\varphi' \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \varphi'(U \cap V) \subset \mathbb{R}^p \times \mathbb{R}^q$ . The coordinates on  $TU$  are  $(x^u, x^{\bar{u}}, y^{(1)u}, y^{(1)\bar{u}})$  and the bundles  $\tau\mathcal{F}_U$  and  $\nu\mathcal{F}_U$  (restricted to  $U$ ), have as coordinates  $(x^u, x^{\bar{u}}, y^{(1)u})$  and  $(x^u, x^{\bar{u}}, y^{(1)\bar{u}})$  respectively.

Let us consider  $x_0 \in U$ . Two curves  $\gamma_{1,2} : I_a = (-a, a) \rightarrow M$ ,  $a > 0$ , have a *transverse contact of order  $r \geq 0$*  in 0 if  $\gamma_1(0) = \gamma_2(0) = x_0$  and if  $r > 0$ , then  $\frac{d^j(\gamma_1 \circ \varphi^{-1})^{\bar{u}}}{dt^j}(0) = \frac{d^j(\gamma_2 \circ \varphi^{-1})^{\bar{u}}}{dt^j}(0)$ ,  $(\forall) \bar{u} = \overline{1, q}$ ,  $j = \overline{1, r}$ ; we denote  $\gamma_1 \overset{r, x_0}{\sim} \gamma_2$ . The „transverse contact of order  $r$ ” relation is an equivalence relation and all the classes  $\hat{\gamma}_{x_0}$ ,  $x_0 \in M$ , give  $\nu^r \mathcal{F}$ , i.e. the *transverse space of order  $r$*  of  $\mathcal{F}$ . The canonical projection  $\pi^r : \nu^r \mathcal{F} \rightarrow M$  gives a local trivial bundle. The generic coordinates  $(x^u, x^{\bar{u}})$  on  $U$  gives rise to some generic coordinates  $(x^u, x^{\bar{u}}, y^{(1)\bar{u}}, \dots, y^{(r)\bar{u}})$  on  $\nu^r \mathcal{F}_U$ , where  $y^{(j)\bar{u}} = \frac{1}{j!} \frac{d^j(\gamma \circ \varphi^{-1})^{\bar{u}}}{dt^j}(0)$  and  $\hat{\gamma}_x$  is a class for  $\overset{r, x}{\sim}$ . Finally one obtain a generic local chart  $(\nu^r \mathcal{F}_U, \nu^r \varphi)$  on  $\nu^r \mathcal{F}$  in a suitable atlas, according to every generic chart  $(U, \varphi)$  on  $M$ , from a suitable atlas. We use notations and constructions according to the non-foliate case, according to [3, Sect. 6.1] and [9].

The generic coordinates  $(x^u, x^{\bar{u}}, y^{(1)\bar{u}}, \dots, y^{(r)\bar{u}})$  on  $\nu^r \mathcal{F}_U$  change according to the rules:

$$\begin{aligned} x^{u'} &= x^u(x^u, x^{\bar{u}}) \\ x^{\bar{u}'} &= x^{\bar{u}}(x^u, x^{\bar{u}}) \\ y^{(1)\bar{u}'} &= y^{(1)\bar{u}} \frac{\partial x^{i'}}{\partial x^i} \\ 2y^{(2)\bar{u}'} &= y^{(1)\bar{u}} \frac{\partial y^{(1)\bar{u}'}}{\partial x^{\bar{u}}} + 2y^{(2)\bar{u}} \frac{\partial y^{(1)\bar{u}'}}{\partial y^{(1)\bar{u}}}, \\ &\vdots \\ ry^{(r)\bar{u}'} &= y^{(1)\bar{u}} \frac{\partial y^{(r-1)\bar{u}'}}{\partial x^{\bar{u}}} + \dots + ry^{(r)\bar{u}} \frac{\partial y^{(r-1)\bar{u}'}}{\partial y^{(r-1)\bar{u}}}. \end{aligned} \tag{1}$$

Denoting  $x^{\bar{u}} = y^{(0)\bar{u}}$ , one have, for  $0 \leq \alpha \leq \beta \leq \gamma \leq r$ :

$$\frac{\partial y^{(\gamma)\bar{u}'}}{\partial y^{(\beta)\bar{u}}} = \frac{\partial y^{(\gamma-\alpha)\bar{u}'}}{\partial y^{(\beta-\alpha)\bar{u}}}.$$

In particular,

$$\frac{\partial y^{(r-1)\bar{u}'}}{\partial y^{(r-1)\bar{u}}} = \frac{\partial x^{\bar{u}'}}{\partial x^{\bar{u}}}.$$

These formulas are true according to similar ones in the non-foliate case [3, Sect. 6.1].

Various bundle structures can be considered over a  $\nu^r \mathcal{F}$ ; for example, for  $0 \leq r' \leq r$ , the canonical projection  $\pi_{r'}^r : \nu^r \mathcal{F} \rightarrow \nu^{r'} \mathcal{F}$  is a foliated bundle. In particular, for  $r \geq 1$ ,  $\pi_{r-1}^r : \nu^r \mathcal{F} \rightarrow \nu^{r-1} \mathcal{F}$  is a (foliated) affine bundle for  $r > 1$  and  $\pi_0^1 : \nu \mathcal{F} \rightarrow \nu^0 \mathcal{F} = M$  is a (foliated) vector bundle (for  $r = 1$ ).

**Proposition 2.1** *For  $1 \leq r' \leq r$ , there is an inclusions of foliated submanifolds (in fact of foliated subbundles over  $M$ ),  $I_{r'}^r : \nu^{r'} \mathcal{F} \rightarrow \nu^r \mathcal{F}$ , where the inclusion*

assigns to an equivalence class in  $[\gamma] \in \nu_{;m}^{r'}\mathcal{F}$  an equivalence class in  $\nu_{;m}^r\mathcal{F}$  that the first  $r - r'$  derivatives vanish, then the next  $r'$  derivatives are the same as the first  $r'$  derivatives of  $\gamma$ .

*Proof.* We use generic coordinates. Denoting  $z^{(\alpha)\bar{u}} = \alpha!y^{(\alpha)\bar{u}}$ ,  $\alpha = \overline{1, r}$  as new coordinates, the local form of  $I_{r'}^r$  is

$$(x^u, x^{\bar{u}}, z^{(1)\bar{u}}, \dots, z^{(r')\bar{u}}) \xrightarrow{I_{r'}^r} (x^u, x^{\bar{u}}, 0, \dots, 0, z^{(1)\bar{u}}, \dots, z^{(r')\bar{u}});$$

using again generic coordinates, the local form of  $I_{r'}^r$  is

$$(x^u, x^{\bar{u}}, y^{(1)\bar{u}}, \dots, y^{(r')\bar{u}}) \xrightarrow{I_{r'}^r} (x^u, x^{\bar{u}}, 0, \dots, 0, \frac{(r - r' + 1)!}{1!}y^{(1)\bar{u}}, \dots, \frac{r!}{r'!}y^{(r')\bar{u}}).$$

Using formulas (1), it easily follows that  $I_{r'}^r$  is globally defined.  $\square$

Let us notice that the local form of  $I_{r-1}^r$  is

$$(x^u, x^{\bar{u}}, y^{(1)\bar{u}}, \dots, y^{(r-1)\bar{u}}) \xrightarrow{I_{r-1}^r} (x^u, x^{\bar{u}}, 0, \frac{2!}{1!}y^{(1)\bar{u}}, \dots, \frac{r!}{(r-1)!}y^{(r-1)\bar{u}})$$

and  $I_r^r$  is the identity of  $\nu^r\mathcal{F}$ .

Thus we have  $I_0^r(M) \subset I_1^r(\nu\mathcal{F}) \subset I_2^r(\nu^2\mathcal{F}) \subset \dots \subset I_{r-1}^r(\nu^{r-1}\mathcal{F}) \subset \nu^r\mathcal{F}$ .

A transverse vector field  $\bar{X} \in \Gamma(\nu\mathcal{F})$  lifts in this way to the transverse section  $I_1^r(\bar{X}) : M \rightarrow \nu^r\mathcal{F}$  of the bundle  $\pi_0^r : \nu^r\mathcal{F} \rightarrow M$ . A more simple case is when  $\bar{X} = \bar{0}$  is the null vector field; its lift is the section  $\bar{0}^r : M \rightarrow \nu^r\mathcal{F}$ ,  $\bar{0}^r(m) = I_0^r(m)$ . Notice that, in the non-foliate case, the lifts constructed here are not the same as the lifts constructed in [8], where the lifts are vector fields.

We denote by  $\mathcal{F}^r$  the foliation on  $\nu^r\mathcal{F}$ . In a similar way as in the non-foliate case in [3, Sect. 6.1], we perform some constructions that are useful later. Notice that in the foliate case the transverse  $\nu\mathcal{F}^r$  play the role of a tangent space for  $\nu^r\mathcal{F}$ , as the tangent space  $\tau T^r M$  is for  $T^r M$  in the non-foliate case in [3].

For every  $r \geq 1$  and  $0 \leq r' \leq r$ , the canonical projection  $\pi_{r'}^r : \nu^r\mathcal{F} \rightarrow \nu^{r'}\mathcal{F}$  induces a transverse map  $\bar{\pi}_{r'}^r : \nu\mathcal{F}^r \rightarrow \nu\mathcal{F}^{r'}$  that is a vector bundle map of foliated vector bundles; notice that  $\pi_0^r = \pi^r$ ,  $\mathcal{F}^0 = \mathcal{F}$ ,  $\nu^1\mathcal{F} = \nu\mathcal{F}$ ,  $\nu^0\mathcal{F} = M$  and  $\bar{\pi}_0^r = \bar{\pi}^r$ . We denote the kernel vector subbundle  $\ker \bar{\pi}_{r'}^r \subset \nu\mathcal{F}^r$  by  $\bar{V}_{r'}^r$ ; it is a foliate vector bundle as well. Since for  $r_1 \leq r_2 \leq r_3$ , one have  $\pi_{r_1}^{r_3} = \pi_{r_2}^{r_3} \circ \pi_{r_1}^{r_2}$  and  $\bar{\pi}_{r_1}^{r_3} = \bar{\pi}_{r_2}^{r_3} \circ \bar{\pi}_{r_1}^{r_2}$ , it follows that there are foliated vector subbundles  $\bar{V}_{r-1}^r \subset \bar{V}_{r-2}^r \subset \dots \subset \bar{V}_0^r \subset \nu\mathcal{F}^r$ . Notice that  $\nu^{r+1}\mathcal{F} \subset \nu\mathcal{F}^r$  is an affine subbundle over  $\nu^r\mathcal{F}$ , for  $r \geq 1$ , while  $\nu^1\mathcal{F} = \nu\mathcal{F}^0 = \nu\mathcal{F}$  for  $r = 0$ .

There is an  $r$ -transverse structures in the fibers of on  $\nu\mathcal{F}^r$ , i.e. a vector bundle map  $J : \nu\mathcal{F}^r \rightarrow \nu\mathcal{F}^r$  (analogous of the  $r$ -tangent structures in the non-foliate case), and its dual  $J^* : \nu^*\mathcal{F}^r \rightarrow \nu^*\mathcal{F}^r$ , given in generic local coordinates by

$$\begin{aligned} J &= \frac{\partial}{\partial y^{(1)\bar{u}}} \otimes \overline{dx^{\bar{u}}} + \frac{\partial}{\partial y^{(2)\bar{u}}} \otimes \overline{dy^{(1)\bar{u}}} + \dots + \frac{\partial}{\partial y^{(r)\bar{u}}} \otimes \overline{dy^{(r-1)\bar{u}}}, \\ J^* &= \overline{dx^{\bar{u}}} \otimes \frac{\partial}{\partial y^{(1)\bar{u}}} + \overline{dy^{(1)\bar{u}}} \otimes \frac{\partial}{\partial y^{(2)\bar{u}}} + \dots + \overline{dy^{(r-1)\bar{u}}} \otimes \frac{\partial}{\partial y^{(r)\bar{u}}}. \end{aligned}$$

A *transverse  $r$ -non-linear connection* is a splitting of  $\nu\mathcal{F}^r$  as a Whitney sum of transverse vector bundles

$$\nu\mathcal{F}^r = \bar{V}_0^r \oplus \bar{H}_0^r, \quad (2)$$

where  $\bar{H}_0^r$  is the  $r$ -horizontal vector bundle, that is canonically isomorphic with  $(\bar{\pi}^r)^*\nu\mathcal{F}$ . We denote by  $h : \nu\mathcal{F}^r \rightarrow \bar{H}_0^r$  the projector given by the above decomposition. Using generic local coordinates, the local form (on the fibers) of the projector  $h$  is

$$(X^{\bar{u}}, Y^{(1)\bar{u}}, \dots, Y^{(r)\bar{u}}) \xrightarrow{h} (X^{\bar{u}} + N_{(1)\bar{v}}^{\bar{u}} Y^{(1)\bar{v}} + \dots + N_{(r)\bar{v}}^{\bar{u}} Y^{(r)\bar{v}}). \quad (3)$$

Given a transverse  $r$ -non-linear connection by a splitting (2), the consecutive images by  $J$  in the fibers of  $\nu\mathcal{F}^r$ ,

$$J(\bar{H}_0^r) = \bar{H}_1^r, \dots, J(\bar{H}_{r-1}^r) = \bar{H}_r^r$$

define some transverse vector subbundles of  $\nu\mathcal{F}^r$ , all isomorphic with  $\bar{H}_0^r$ , such that there are the following Whitney sum decompositions

$$\bar{V}_0^r = \bar{H}_1^r \oplus \dots \oplus \bar{H}_r^r, \quad \nu\mathcal{F}^r = \bar{H}_0^r \oplus \bar{H}_1^r \oplus \dots \oplus \bar{H}_r^r. \quad (4)$$

Notice that  $\bar{H}_r^r = \bar{V}_{r-1}^r$  and we can prove the following result.

**Proposition 2.1** *Any splitting  $\nu\mathcal{F}^r = \bar{V}_{r-1}^r \oplus \bar{H}_{r-1}^r$  gives rise to a splitting (2).*

*Proof.* The given splitting gives rise also to a splitting  $\nu^*\mathcal{F}^r = (\bar{V}_{r-1}^r)^* \oplus (\bar{H}_{r-1}^r)^*$  and denote  $\bar{W}^r \stackrel{\text{not}}{=} (\bar{V}_{r-1}^r)^* \subset \nu^*\mathcal{F}^r$ . Let us consider the consecutive images by  $J^*$  in the fibers of  $\nu^*\mathcal{F}^r$ ,

$$J^*(\bar{W}^r) = \bar{W}^{r-1}, \dots, J^*(\bar{W}^1) = \bar{W}^0$$

defining some transverse vector subbundles of  $\nu^*\mathcal{F}^r$ , all isomorphic with  $(\bar{V}_{r-1}^r)^*$  and where  $\bar{W}^0 = (\bar{V}_0^r)^\circ$  is the polar (or the annihilator). Considering  $\bar{W} = \bar{W}^1 \oplus \dots \oplus \bar{W}^r$ , we have  $\nu^*\mathcal{F}^r = (\bar{V}_0^r)^\circ \oplus \bar{W}$ , thus  $\bar{W}$  is isomorphic with  $(\bar{V}_0^r)^*$ . It follows that the inclusion  $(\bar{V}_0^r)^* \subset \nu^*\mathcal{F}^r$  reverses by duality to a transverse epimorphism (in fact a projector)  $\Pi : \nu\mathcal{F}^r \rightarrow \bar{V}_0^r$  that gives a splitting (2), where  $\bar{H}_0^r = \ker \Pi$ .  $\square$

Using generic local coordinates, the local form (on the fibers) of the projector  $\Pi : \nu\mathcal{F}^r \rightarrow \bar{V}_{r-1}^r$  is

$$(X^{\bar{u}}, Y^{(1)\bar{u}}, \dots, Y^{(r)\bar{u}}) \xrightarrow{\Pi} (M_{(0)\bar{v}}^{\bar{u}} X^{\bar{v}} + M_{(1)\bar{v}}^{\bar{u}} Y^{(1)\bar{v}} \dots + M_{(r-1)\bar{v}}^{\bar{u}} Y^{(r-1)\bar{v}} + Y^{(r)\bar{u}}). \quad (5)$$

The local coefficients  $N$  and  $M$  in formulas (3) and (5) respectively are called as *dual coefficients* in [3, sect. 6.6].

A *transverse  $r$ -semi-spray* is a foliate section  $S : \nu^r \mathcal{F} \rightarrow \nu^{r+1} \mathcal{F}$  of the affine bundle  $\pi_r^{r+1} : \nu^{r+1} \mathcal{F} \rightarrow \nu^r \mathcal{F}$ . Since  $\nu^{r+1} \mathcal{F} \subset \nu \mathcal{F}^r$ , it follows that an  $r$ -semi-spray can be regarded as well as a transverse section  $S : \nu^r \mathcal{F} \rightarrow \nu \mathcal{F}^r$ ; using generic local coordinates, the local form of  $S : \nu^r \mathcal{F} \rightarrow \nu^{r+1} \mathcal{F}$  is

$$(x^u, x^{\bar{u}}, y^{(1)\bar{u}}, \dots, y^{(r)\bar{u}}) \xrightarrow{S} (x^u, x^{\bar{u}}, y^{(1)\bar{u}}, \dots, ry^{(r)\bar{u}}, (r+1)S^{\bar{u}}(x^{\bar{u}}, y^{(1)\bar{u}}, \dots, y^{(r)\bar{u}})). \quad (6)$$

**Proposition 2.2** *Any transverse  $r$ -semi-spray gives rise to a transverse  $r$ -non-linear connection, i.e. a splitting (2).*

*Proof.* If  $\bar{X} \in \Gamma(\nu \mathcal{F}^r)$  is a transverse section, let us denote by  $L_{\bar{X}}$  the transverse Lie derivation induced in the transverse tensor algebra. As in the non-foliate case in [2], the operators

$$h = \frac{1}{k+1} (kI - L_S J), v = \frac{1}{k+1} (I + L_S J) \quad (7)$$

are complementary projectors on the fibers of the transverse bundle  $\nu \mathcal{F}^r \rightarrow \nu^r \mathcal{F}$  and  $v(\nu \mathcal{F}^r) = \bar{V}_{r-1}^r$ . Using Proposition 2.1, a transverse  $r$ -non-linear connection is obtained.  $\square$

Using generic coordinates, it can be proved using (7), that the dual coefficients (5) of the non-linear connection are given by the formulas

$$M_{(r)\bar{v}}^{\bar{u}} = -\frac{\partial S^{\bar{u}}}{\partial y^{(1)\bar{v}}}, M_{(r-1)\bar{v}}^{\bar{u}} = -\frac{\partial S^{\bar{u}}}{\partial y^{(2)\bar{v}}}, M_{(1)\bar{v}}^{\bar{u}} = -\frac{\partial S^{\bar{u}}}{\partial y^{(r)\bar{v}}}. \quad (8)$$

A fact that we use latter is the following result.

**Proposition 2.3** *A transverse  $r$ -non-linear connection and a transverse riemannian metric in the fibers of  $\bar{V}_{r-1}^r$  lifts to a transverse riemannian metric on  $\nu \mathcal{F}^r$ . Conversely, a transverse riemannian metric on  $\nu \mathcal{F}^r$  gives a transverse  $r$ -non-linear connection and a transverse riemannian metric in the fibers of  $\bar{V}_{r-1}^r$ .*

*Proof.* Every riemannian metric in the fibers of  $\bar{V}_{r-1}^r \cong \bar{H}_0^r \cong (\bar{\pi}_0^r)^* \nu \mathcal{F}$  lifts to transverse riemannian metrics on  $\bar{H}_1^r, \dots, \bar{H}_r^r$  and consequently to a riemannian metric on  $\nu \mathcal{F}^r$ , that becomes a riemannian foliation. Conversely, if  $\mathcal{F}^r$  is a riemannian foliation, then a transverse riemannian metric in the fibers of  $\nu \mathcal{F}^r$  gives a decomposition (2), where  $\bar{H}_0^r = (\bar{V}_0^r)^\perp$  and induces a transverse riemannian metric in the fibers of the vector subbundle  $\bar{H}_0^r$ .  $\square$

The  $r$ -transverse non-linear connections, semi-sprays and riemannian metrics are involved in the case of regular  $r$ -transverse lagrangians that we consider in the sequel.

An  *$r$ -transverse lagrangian* (a transverse lagrangian of order  $r \geq 1$ , i.e. locally projectable on an  $r$ -lagrangian) is a continuous real map  $L : \nu^r \mathcal{F} \rightarrow \mathbb{R}$ ,

smooth on an open fibered submanifold  $\nu_*^r \mathcal{F} \subset \nu^r \mathcal{F}$ . The cases studied in the paper are when  $\nu_*^r \mathcal{F} = \nu^r \mathcal{F}$ , i.e.  $L$  is smooth, or when  $\nu^r \mathcal{F} \setminus \nu_*^r \mathcal{F}$  contains  $I_{r-1}^r(\nu^{r-1} \mathcal{F})$ , i.e.  $L$  is slashed. For sake of simplicity, we perform the next constructions in the case of a smooth  $L$ , in the slashed case we must be care of domains where the objects are defined. The *vertical hessian* of  $L$  is the bilinear form  $h$  in the fibers of  $\bar{V}_{r-1}^r$ , given in some generic coordinates by

$$h_{\bar{u}\bar{v}} = \frac{\partial^2 L}{\partial y^{(r)\bar{u}} \partial y^{(r)\bar{v}}}.$$

We say that  $L$  is *regular* if its vertical hessian is non-degenerated. The fibers of the fibered manifold  $\nu^r \mathcal{F} \rightarrow \nu^{r-1} \mathcal{F}$  are affine spaces. Using (1), the generic coordinates on fibers change according the formulas

$$ry^{(r)\bar{u}'} = \Gamma(y^{(r-1)\bar{u}'}) + ry^{(r)\bar{u}} \frac{\partial x^{\bar{u}'}}{\partial x^{\bar{u}}},$$

where

$$\Gamma = y^{(1)\bar{u}} \frac{\partial}{\partial x^{\bar{u}}} + 2y^{(2)\bar{u}} \frac{\partial}{\partial y^{(1)\bar{u}}} + \dots + ry^{(r)\bar{u}} \frac{\partial}{\partial y^{(r-1)\bar{u}}}.$$

**Proposition 2.4** *a) If an  $r$ -lagrangian  $L$  is regular, then it can define canonically a transverse  $r$ -semi-spray and a transverse  $r$ -non- linear connection.*

*b) If the vertical hessian of an  $r$ -lagrangian  $L$  is positively defined, then  $\mathcal{F}^r$  is a riemannian foliation.*

*Proof.* In order to prove a) it suffices to construct a transverse  $r$ -semi-spray. Then a transverse  $r$ -non- linear connection can be constructed by Proposition 2.2. We use generic local coordinates, according to formula (6). As in the non-foliate case [3, Theorem 8.8.1] (see also [1]), the local form of the functions  $S^{\bar{u}}$  can be taken according to the formula

$$S^{\bar{u}} = \frac{1}{2(r+1)} h^{\bar{u}\bar{v}} \left( \Gamma \left( \frac{\partial L}{\partial y^{(r)\bar{v}}} \right) - \frac{\partial L}{\partial y^{(r-1)\bar{v}}} \right),$$

where  $(h^{\bar{u}\bar{v}}) = (h_{\bar{u}\bar{v}})^{-1}$ . In order to prove b), one construct a transverse riemannian metric  $H$  in the fibers of  $\nu^r \mathcal{F}$ , using its decomposition given by formula (4) and taking into account that all the summands are isomorphic with  $V_{r-1}^r$ , where  $h$  is a riemannian metric on fibers.  $\square$

According to the case of trivial foliation of  $M$  by points in [9],  $\nu^{r-1} \mathcal{F} \times_M \nu^* \mathcal{F} \stackrel{\text{not.}}{=} \nu^{r*} \mathcal{F}$  play the role of the vectorial dual of the affine bundle  $\nu^r \mathcal{F} \rightarrow \nu^{r-1} \mathcal{F}$ . The usual partial derivatives of  $L$  in the highest order transverse coordinates define a well-defined Legendre map  $\mathcal{L} : \nu^r \rightarrow \nu^{r*} \mathcal{F}$ , i.e.

$$(x^u, x^{\bar{u}}, y^{(1)\bar{u}}, \dots, y^{(r)\bar{u}}) \xrightarrow{\mathcal{L}} (x^u, x^{\bar{u}}, y^{(1)\bar{u}}, \dots, y^{(r-1)\bar{u}}, \frac{\partial L}{\partial y^{(r)\bar{u}}}).$$



If  $L$  is regular, then  $\mathcal{L}$  is a local diffeomorphism. If  $\mathcal{L}$  is a global diffeomorphism we say that  $L$  is *hyperregular*. We say that  $H : \nu^{r*}\mathcal{F} \rightarrow \mathbb{R}$ ,  $H = L \circ \mathcal{L}^{-1}$  is the pseudo-hamiltonian associated with  $L$ . If  $\mathcal{L}^{-1}$  has the local form

$$\mathcal{L}^{-1}(x^u, x^{\bar{u}}, y^{(1)\bar{u}}, \dots, y^{(r-1)\bar{u}}, p_{\bar{u}}) = (x^u, x^{\bar{u}}, y^{(1)\bar{u}}, \dots, y^{(r-1)\bar{u}}, H^{\bar{u}}(x^u, x^{\bar{u}}, y^{(1)\bar{u}}, \dots, y^{(r-1)\bar{u}}, p_{\bar{u}})),$$

then  $H$  has the local form

$$H(x^u, x^{\bar{u}}, y^{(1)\bar{u}}, \dots, y^{(r-1)\bar{u}}, p_{\bar{u}}) = L(x^u, x^{\bar{u}}, y^{(1)\bar{u}}, \dots, y^{(r-1)\bar{u}}, H^{\bar{u}}(x^u, x^{\bar{u}}, y^{(1)\bar{u}}, \dots, y^{(r-1)\bar{u}}, p_{\bar{u}})).$$

For  $0 \leq r' \leq r$ , let us denote  $\nu^{r', (r-r')*}\mathcal{F} = \nu^{r'}\mathcal{F} \times_M (\nu^*\mathcal{F})^{r-r'}$ , where  $(\nu^*\mathcal{F})^{r-r'} = \nu^*\mathcal{F} \times_M \dots \times_M \nu^*\mathcal{F}$ , with the fibered product of  $(r-r')$ -times. In particular,  $\nu^{r*} = \nu^{r-1, r*}\mathcal{F} = \nu^{r-1}\mathcal{F} \times_M \nu^*\mathcal{F}$ .

A *transverse slashed lagrangian* of order  $r$  is a continuous map  $L^r : \nu^r\mathcal{F} \rightarrow \mathbb{R}$  that is differentiable on an open fibered submanifold  $\nu_*^r\mathcal{F} \subset \nu^r\mathcal{F}$ , called a *slashed bundle*. All the above constructions can be adapted for slashed lagrangians.

Let us suppose that  $L^r$  is *hyperregular*, i.e. the Legendre map  $\mathcal{L}^{(r)} : \nu_*^r\mathcal{F} \rightarrow \nu^{1, (r-1)*}\mathcal{F} = \nu^{r-1}\mathcal{F} \times_M \nu^*\mathcal{F}$  is a diffeomorphism on its image. Let us suppose also that  $\mathcal{L}^{(r)}(\nu_*^r) = \nu_*^{1, (r-1)*}\mathcal{F} = \nu_*^{r-1}\mathcal{F} \times_M \nu_*^*\mathcal{F}$ ; here  $\nu_*^*\mathcal{F} = \nu^*\mathcal{F} \setminus \{\bar{0}\}$  (where  $\{\bar{0}\}$  is the image of the section obtained by all velocities vanishing, can be identified with  $M$ ) and  $\nu_*^{r-1}\mathcal{F}$  is a slashed subbundle of  $\nu^{r-1}\mathcal{F}$ . We denote by  $H^{1, r-1} = L^r \circ (\mathcal{L}^{(r)})^{-1} : \nu_*^{1, (r-1)*}\mathcal{F} \rightarrow \mathbb{R}$  its pseudo-hamiltonian. (See [9] for its classical definition and [11] for a coordinate description of the whole construction in the non-foliate case). Analogous, for  $0 \leq j < r-1$ , we suppose, step by step, backward from  $r-1$  from 0, that there the usual partial derivatives of  $L^{(j+1)} : \nu_*^{j+1, (r-j-1)*}\mathcal{F} = \nu_*^{r-j-1}\mathcal{F} \times_M (\nu_*^*\mathcal{F})^{j+1} \rightarrow \mathbb{R}$  in the highest order transverse coordinates (of order  $j+1$ ) define a well-defined Legendre map  $\mathcal{L}^{(j+1)} : \nu_*^{j+1, (r-j-1)*}\mathcal{F} = \nu_*^{j+1}\mathcal{F} \times_M (\nu_*^*\mathcal{F})^{r-j-1} \rightarrow \nu^{j, (r-j)*}\mathcal{F} = \nu^j\mathcal{F} \times_M (\nu^*\mathcal{F})^{r-j}$ . We suppose that  $\mathcal{L}^{(j+1)}$  is a diffeomorphism on its image and the image is exactly  $\mathcal{L}^{(j+1)}(\nu_*^{j+1, (r-j-1)*}\mathcal{F}) = \nu_*^{j, (r-j)*}\mathcal{F} = \nu_*^j\mathcal{F} \times_M (\nu_*^*\mathcal{F})^{r-j}$ .

Then the pseudo-hamiltonian  $L^{(j)} = L^{(j+1)} \circ (\mathcal{L}^{(j+1)})^{-1} : \nu_*^{j, (r-j)*}\mathcal{F} \rightarrow \mathbb{R}$  can be considered. Finally, for  $j = 0$ , we obtain a transverse slashed lagrangian  $L^{(0)} = L^1 \circ (\mathcal{L}^{(1)})^{-1} : \nu_*^{0, r*}\mathcal{F} = (\nu_*^*\mathcal{F})^r \rightarrow \mathbb{R}$  and we suppose that  $\mathcal{L}^{(1)} : \nu_*^{1, (r-1)*}\mathcal{F} = \nu_*^*\mathcal{F} \times_M (\nu_*^*\mathcal{F})^{r-1} \rightarrow \nu_*^{0, r*}\mathcal{F} = (\nu_*^*\mathcal{F})^r \subset \nu^{0, r*}\mathcal{F} = (\nu^*\mathcal{F})^r$  is a diffeomorphism. It follows a diffeomorphism  $\mathcal{L} = \mathcal{L}^{(1)} \circ \dots \circ \mathcal{L}^{(r)} : \nu_*^r\mathcal{F} \rightarrow (\nu_*^*\mathcal{F})^r$  and a transverse slashed lagrangian  $L^{(0)} : (\nu_*^*\mathcal{F})^r \rightarrow \mathbb{R}$ . The canonical diagonal inclusion  $\nu^*\mathcal{F} \rightarrow (\nu^*\mathcal{F})^r$  sends  $\nu_*^*\mathcal{F} \rightarrow (\nu_*^*\mathcal{F})^r$ . We suppose that the restriction of  $L^{(0)}$  to the diagonal is a positively admissible lagrangian on  $\nu^*\mathcal{F}$ , in fact a transverse hamiltonian  $H : \nu_*^*\mathcal{F} \rightarrow \mathbb{R}$ . If the given transverse lagrangian  $L^r : \nu^r\mathcal{F} \rightarrow \mathbb{R}$  fulfills all the above conditions, we say that  $L$  itself is a *positively admissible lagrangian* (of order  $r$ ) and  $H$  is its *diagonal hamiltonian*.

The existence of a lifted metric, from the base space to the higher order tangent bundle, is an well-known fact in the non-foliate case (see, for example

[3, Sect. 9.2]); we have to consider a simpler construction in the foliated case, that it is also vertically exact, as in [9, 11].

**Proposition 2.5** *Any transverse metric  $g$  on  $\nu F$  gives canonically a positively admissible lagrangian  $L^{(r)}$  of order  $r$  and a canonical vertically exact transverse riemannian metric  $g^{(r)}$  on  $\nu^r \mathcal{F}$ , for any  $r \geq 1$ .*

*Proof.* We proceed by induction over  $r \geq 1$ . We consider the quadratic first order lagrangian  $L^{(1)} : \nu \mathcal{F} \rightarrow \mathbb{R}$ ,  $L^{(1)}(x, y^{(1)}) = g_x(y^{(1)}, y^{(1)})$ . The Levi-Civita connection of the transverse metric  $g$  on  $\nu F$  gives rise to the geodesic first order spray  $S^{(1)} : \nu \mathcal{F} \rightarrow \nu^2 \mathcal{F}$  of  $L^{(1)}$  and to a second order lagrangian  $L^{(2)} : \nu^2 \mathcal{F} \rightarrow \mathbb{R}$ ,  $L^{(2)}(x, y^{(1)}, y^{(2)}) = L^{(1)}(x, y^{(1)}) + L^{(1)}(x, y^{(2)} - S^{(1)}(x, y^{(1)}))$ . Assume that  $L^{(r-1)} : \nu^{r-1} \mathcal{F} \rightarrow \mathbb{R}$  has been constructed. An  $(r-1)$ -order spray  $S^{(r)} : \nu^r \mathcal{F} \rightarrow \nu^{r+1} \mathcal{F}$  can be constructed according to Proposition 2.4, since  $L^{(r-1)}$  is  $r$ -regular. It follows  $L^{(r)} : \nu^r \mathcal{F} \rightarrow \mathbb{R}$ ,  $L^{(r)}(x, y^{(1)}, \dots, y^{(r)}) = L^{(r-1)}(x, y^{(1)}, \dots, y^{(r-1)}) + L^{(1)}(x, y^{(r)} - S^{(r-1)})$ , that is a positively admissible lagrangian of order  $r$ ; the diagonal hamiltonian of  $L^{(r)}$  is just the dual hamiltonian of  $L^{(1)}$ . According to Proposition 2.3, the lagrangian  $L^{(r)}$  gives rise to a transverse riemannian metric in the fibers of  $\nu^r \mathcal{F}$ , that is vertically exact.  $\square$

### 3 The main results

We can state and prove the main results of the paper.

**Theorem 3.1** *The lifted foliation  $\mathcal{F}^r$  on  $\nu^r \mathcal{F}$  is riemannian for some  $r \geq 1$  iff  $\mathcal{F}$  is a riemannian foliation.*

*Proof.* The sufficiency is well-known and it follows by Proposition 2.5. The necessity follows considering the submanifold inclusion  $I_0^r(M) \subset \nu^r \mathcal{F}$ . The induced transverse riemannian metric on the foliation induced on  $I_0^r(M)$  gives a transverse riemannian metric on  $\nu \mathcal{F}$ , thus  $\mathcal{F}$  is riemannian.  $\square$

We say that a foliation  $\mathcal{F}$  is transversally almost parallelizable if there is a  $\mathcal{F}$ -transverse vector bundle  $\xi$  over  $M$ , such that  $\xi \oplus \nu \mathcal{F}$  is transversally parallelizable. If a foliation  $\mathcal{F}$  is transversally parallelizable, then it is a riemannian one by a transverse metric given by a parallelization. In the case of an almost transversally parallelizable, any transverse riemannian metric given by a parallelism of  $\xi \oplus \nu \mathcal{F}$  induces a transverse riemannian metric on  $\nu \mathcal{F}$ . Thus the following statement holds true.

**Corollary 3.1** *If the lifted foliation  $\mathcal{F}^r$  on  $\nu^r \mathcal{F}$  is transversally parallelizable or almost parallelizable for some  $r \geq 1$ , then  $\mathcal{F}$  is a riemannian foliation.*

The proof of Theorem 3.1 can not give any answer to the following question: *when is  $\mathcal{F}$  riemannian if the foliation induced on  $\nu^r \mathcal{F} \setminus I_{r-1}^r(\nu^{r-1} \mathcal{F})$  is riemannian for some  $r \geq 1$ ?* We are going to relate this question to the existence of a

certain transverse slashed lagrangian  $L^r$  of order  $r$ , asking that the open subset  $\nu_*^r \mathcal{F} \subset \nu^r \mathcal{F}$  that does not contains  $I_{r-1}^r(\nu^{r-1} \mathcal{F})$ . We say that a such lagrangian  $L^r$  is  $r$ -regular if its vertical hessian, according to the induced affine bundle structure  $\pi_{r-1}^r : \nu^r \mathcal{F} \rightarrow \nu^{r-1} \mathcal{F}$ , is non-degenerate. In order to give an answer to the above question, we consider below some other regularity conditions for slashed lagrangians of order  $r$ .

A transverse bundle of order  $r$ ,  $\nu^r \mathcal{F}$  can be regarded as a fibered manifold  $\pi_{r'}^r : \nu^r \mathcal{F} \rightarrow \nu^{r'} \mathcal{F}$ ,  $(\forall) 0 \leq r' < r$ . We denote  $\nu^{r', (r-r')*} \mathcal{F} = \nu^{r'} \mathcal{F} \times_M (\nu^* \mathcal{F})^{r-r'}$  (where  $(\nu^* \mathcal{F})^{r-r'} = \nu^* \mathcal{F} \times_M \cdots \times_M \nu^* \mathcal{F}$ , with the fibered product of  $(r-r')$ -times and  $\nu^* \mathcal{F}$  is the transverse bundle dual to  $\nu \mathcal{F}$ ).

In particular, according to the case of trivial foliation of  $M$  by points in [9],  $\nu^{1, (r-1)*} \mathcal{F} = \nu^{r-1} \mathcal{F} \times_M \nu^* \mathcal{F}$  is denoted by  $\nu^{r*} M$  and play the role of the vectorial dual of the affine bundle  $\nu^r \mathcal{F} \rightarrow \nu^{r-1} \mathcal{F}$ .

A transverse slashed lagrangian of order  $r$  is a map  $L^r : \nu^r \mathcal{F} \rightarrow \mathbb{R}$  that is differentiable on an open subset  $\nu_*^r \mathcal{F} \subset \nu^r \mathcal{F}$ , where  $\nu^r \mathcal{F} \setminus \nu_*^r \mathcal{F}$  contains  $I_{r-1}^r(\nu^{r-1} \mathcal{F})$ .

We denote by  $\nu_*^{r'} \mathcal{F} = \pi_{r'}^r(\nu_*^r \mathcal{F}) \subset \nu^{r'} \mathcal{F}$  and we consider the slashed bundles  $\nu_*^r \mathcal{F} = \nu^* \mathcal{F} \setminus \{0\}$  and  $\nu_*^{r', (r-r')*} \mathcal{F} = \nu^{r'} \mathcal{F} \times_M (\nu_*^* \mathcal{F})^{r-r'}$ , for  $0 \leq r' \leq r-1$ . The elements of a fiber  $\left( \nu_*^{r', (r-r')*} \mathcal{F} \right)_{;m}$  of  $\nu_*^{r', (r-r')*} \mathcal{F} \rightarrow M$  are couples of higher order elements and  $(r-r')$  first order momenta. The usual partial derivatives of  $L$  in the highest order transverse coordinates define a well-defined Legendre map  $\mathcal{L}^{(r-1)} : \nu_*^r \rightarrow \nu^{1, (r-1)*} \mathcal{F} = \nu^{r-1} \mathcal{F} \times_M \nu^* \mathcal{F}$ . We suppose first that  $\mathcal{L}^{(r)}$  is a diffeomorphism on its image and the image is exactly  $\mathcal{L}^{(r-1)}(\nu_*^r) = \nu_*^{1, (r-1)*} \mathcal{F} = \nu_*^{r-1} \mathcal{F} \times_M \nu^* \mathcal{F}$ . Then the energy  $L^{(r-1)} : \nu_*^{1, (r-1)*} \mathcal{F} \rightarrow \mathbb{R}$  of the dual affine hamiltonian of  $L$  (see [9] for its classical definition and [11] for a coordinate description of the whole construction in the non-foliate case). Analogous, for  $0 \leq j < r-1$ , we suppose, step by step, backward from  $r-1$  from 0, that there the usual partial derivatives of  $L^{(j+1)} : \nu_*^{j+1, (r-j-1)*} \mathcal{F} = \nu_*^{r-j-1} \mathcal{F} \times_M (\nu_*^* \mathcal{F})^{j+1} \rightarrow \mathbb{R}$  in the highest order transverse coordinates (of order  $j+1$ ) define a well-defined Legendre map  $\mathcal{L}^{(j+1)} : \nu_*^{j+1, (r-j-1)*} \mathcal{F} = \nu_*^{j+1} \mathcal{F} \times_M (\nu_*^* \mathcal{F})^{r-j-1} \rightarrow \nu^{j, (r-j)*} \mathcal{F} = \nu^j \mathcal{F} \times_M (\nu^* \mathcal{F})^{r-j}$ . We suppose that  $\mathcal{L}^{(j+1)}$  is a diffeomorphism on its image and the image is exactly  $\mathcal{L}^{(j+1)}(\nu_*^{j+1, (r-j-1)*} \mathcal{F}) = \nu_*^{j, (r-j)*} \mathcal{F} = \nu_*^j \mathcal{F} \times_M (\nu_*^* \mathcal{F})^{r-j}$ . Then the energy  $L^{(j)} : \nu_*^{j, (r-j)*} \mathcal{F} \rightarrow \mathbb{R}$  of the dual affine hamiltonian of  $L^{(j+1)}$  can be considered. Finally, for  $j=0$ , we obtain a transverse lagrangian  $L^{(0)} : \nu_*^{0, r*} \mathcal{F} = (\nu_*^* \mathcal{F})^r \rightarrow \mathbb{R}$  as the energy of the dual affine hamiltonian  $L^{(1)}$  and we suppose that  $\mathcal{L}^{(1)} : \nu_*^{1, (r-1)*} \mathcal{F} = \nu_*^* \mathcal{F} \times_M (\nu_*^* \mathcal{F})^{r-1} \rightarrow \nu_*^{0, r*} \mathcal{F} = (\nu_*^* \mathcal{F})^r \subset \nu^{0, r*} \mathcal{F} = (\nu^* \mathcal{F})^r$  is a diffeomorphism. It follows a diffeomorphism  $\mathcal{L} = \mathcal{L}^{(1)} \circ \cdots \circ \mathcal{L}^{(r)} : \nu_*^r \rightarrow (\nu_*^* \mathcal{F})^r$  and a transverse lagrangian  $L^{(0)} : (\nu_*^* \mathcal{F})^r \rightarrow \mathbb{R}$ . The canonical diagonal inclusion  $\nu^* \mathcal{F} \rightarrow (\nu^* \mathcal{F})^r$  sends  $\nu_*^* \mathcal{F} \rightarrow (\nu_*^* \mathcal{F})^r$ . We suppose that the restriction of  $L^{(0)}$  to the diagonal is a positively admissible lagrangian on  $\nu^* \mathcal{F}$ , in fact a transverse hamiltonian  $H : \nu^* \mathcal{F} \rightarrow \mathbb{R}$ . If the given transverse lagrangian  $L^r : \nu^r \mathcal{F} \rightarrow \mathbb{R}$  fulfills all the

above conditions, we say that  $L$  itself is a *positively admissible lagrangian* (of order  $r$ ) and  $H$  is its *diagonal hamiltonian*.

The vertical bundle  $V$  of the affine bundle  $\pi_{r-1}^r : \nu^r \mathcal{F} \rightarrow \nu^{r-1} \mathcal{F}$  has the form  $(\pi_0^r)^* \nu \mathcal{F} \rightarrow \nu^r \mathcal{F}$ . We say that a transverse metric  $g$  on  $\nu^r \mathcal{F}$  is *vertically exact* if there is a positively admissible lagrangian of order  $r$ ,  $L : \nu^r \mathcal{F} \rightarrow \mathbb{R}$  such that the restriction of  $g$  to  $V$  is the same as the vertical hessian  $\text{Hess } L$ . We say in this case that the riemannian foliation  $\mathcal{F}^r$  is *vertically exact*. These definitions can be easily adapted for the case of slashed bundles  $\nu_*^r \mathcal{F}$ .

The main technical tool to prove the necessity of Theorems 3.2 and 3.3 below is the following result proved in [12, Proposition 2.2].

**Proposition 3.1** *Let  $p_1 : E_1 \rightarrow M$  and  $p_2 : E_2 \rightarrow M$  be foliated vector bundles over a foliated manifold  $(M, \mathcal{F})$  and  $q_2 : E_{2*} \rightarrow M$  be the slashed bundle. If there are a positively admissible lagrangian  $L : E_2 \rightarrow \mathbb{R}$  and a metric  $b$  on the pull back bundle  $q_2^* E_1 \rightarrow E_{2*}$ , foliated with respect to  $\mathcal{F}_{E_{2*}}$ , then there is a foliated metric on  $E_1$ , with respect to  $\mathcal{F}$ .*

We can now state and prove the following Theorems.

**Theorem 3.2** *Let  $\mathcal{F}$  be a foliation on a manifold  $M$  and  $\mathcal{F}_0^r$  be the lifted foliation in a suitable slashed bundle  $\nu_*^r \mathcal{F}$  of the  $r$ -normal bundle  $\nu^r \mathcal{F}$ . Then  $\mathcal{F}_0^r$  is riemannian and vertically exact for some  $r \geq 1$  iff  $\mathcal{F}$  is riemannian.*

In particular, it follows that any transverse metric  $g$  on  $\nu^r \mathcal{F}$  gives rise to a canonical lagrangian on  $\nu^r \mathcal{F}$ , coming from the vertical part of the vertically exact transverse riemannian metric on  $\nu^r \mathcal{F}$ . So, it is natural to ask that only the existence of a lagrangian on  $\nu^r \mathcal{F}$  guaranties that  $\mathcal{F}$  is riemannian. One have a positive answer, as follows.

**Theorem 3.3** *If  $(M, \mathcal{F})$  is a foliated manifold, then there is a positively admissible lagrangian on  $\nu^r \mathcal{F}$  for some  $r \geq 1$  iff the foliation  $\mathcal{F}$  is riemannian.*

*Proof (of Theorems 3.2 and 3.3).* The sufficiency for both Theorems follow by Proposition 2.5. The necessity for Theorem 3.3 follows using Proposition 3.1 with  $E_1 = E_2 = \nu^* \mathcal{F}$  and  $H$  the diagonal hamiltonian. Finally, the necessity for Theorem 3.2 follows thanks to Theorem 3.3 for the positively admissible lagrangian on  $\nu^r \mathcal{F}$ , given by the condition that the riemannian metric on  $\nu^r \mathcal{F}$  is vertically exact.  $\square$

Finally, as in [12], the following question arises: *can we drop in Theorem 3.2 the condition that  $\mathcal{F}_0^r$  is vertically exact?*

## References

- [1] Bucataru I., *Canonical semisprays for higher order Lagrange spaces*, C. R. Acad. Sci. Paris, Ser. I 345 (2007) 269–272.

- [2] Crampin M., Sarlet W., Cantrijn F., *Higher Order differential equations and higher order lagrangian mechanics*, Math. Proc. Camb. Phil. Soc., 86 (1986), 565–587.
- [3] R. Miron, *The Geometry of Higher-Order Lagrange Spaces. Applications to Mechanics and Physics*, FTPH no. 82, Kluwer Academic Publisher, 1997.
- [4] Józefowicz M., Wolak R., *Finsler foliations of compact manifolds are Riemannian*, Differential Geometry and its Applications, 26 (2) (2008) 224–226.
- [5] Miernowski A., Mozgawa W., *Lift of the Finsler foliation to its normal bundle*, Differential Geometry and its Applications, 24 (2006) 209–214.
- [6] Molino P., *Riemannian foliations*, Progress in Mathematics, Vol. 73, Birkhäuser, Boston, 1988.
- [7] Morimoto A., *Prolongations of G-structure to tangent bundles of higher order*, Nagoya Math. J., 38 (1970) 153–179.
- [8] Morimoto, A., *Liftings of tensor fields and connections to tangent bundles of higher order*, Nagoya Math. J., 40 (1970) 99–120.
- [9] Popescu P., Popescu M., *Affine Hamiltonians in higher order geometry*, International Journal of Theoretical Physics, 46 (10) (2007) 2531–2549.
- [10] Popescu P., Popescu M., *Lagrangians adapted to submersions and foliations*, Differential Geometry and its Applications, 27 (2) (2009) 171–178.
- [11] Popescu M., Popescu P., *Lagrangians and higher order tangent spaces*, Balkan Journal of Geometry and its Applications, 15 (1) (2010) 142–148.
- [12] Popescu P., Popescu M., *Foliated vector bundles and Riemannian foliations*, C. R. Acad. Sci. Paris, Ser. I 349 (2011) 445–449.
- [13] Tarquini C., *Feuilletages de type fini compact*, C. R. Acad. Sci. Paris, Ser. I, 339 (2004) 209–214.
- [14] Vaisman I., *Hamiltonian structures on foliations*, J. Math. Phys., 43, 10 (2002), 4966–4977.
- [15] Wolak R.A., *On transverse structures of foliations*, Proceedings of the 13th winter school on abstract analysis (Srní, 1985). Rend. Circ. Mat. Palermo (2) Suppl. No. 9 (1985), 227–243.
- [16] Wolak R.A., *Leaves of foliations with a transverse geometric structure of finite type*, Publ. Mat. 33 (1989), no. 1, 153–162
- [17] Wolak R.A., *Foliated and associated geometric structures on foliated manifolds*, Ann. Fac. Sci. Toulouse, V. Sér., Math., 10 (3) (1989) 337–360.